

Mathematical Science I Ken-Ichi Nakamura (Room 386)
Homework 4 (Due: June 8)

Problem 1 Let $C([0, 1])$ be the set of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ and define

$$d_1(f, g) := \int_0^1 |f(x) - g(x)| dx$$

for $f, g \in C([0, 1])$. Show that $(C([0, 1]), d_1)$ is not complete.

Hint: Consider the sequence $\{f_n\}_{n \geq 3}$ defined by

$$f_n(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} - \frac{1}{n}, \\ n(x + \frac{1}{n} - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Problem 2 Prove that every sequentially compact subset of a metric space is bounded and closed.

Problem 3 (Extra) Prove that the notions of sequentially compactness and compactness are equivalent in a metric space.